# Supplemental Appendices: Influencer Authenticity: To Grow or To Monetize

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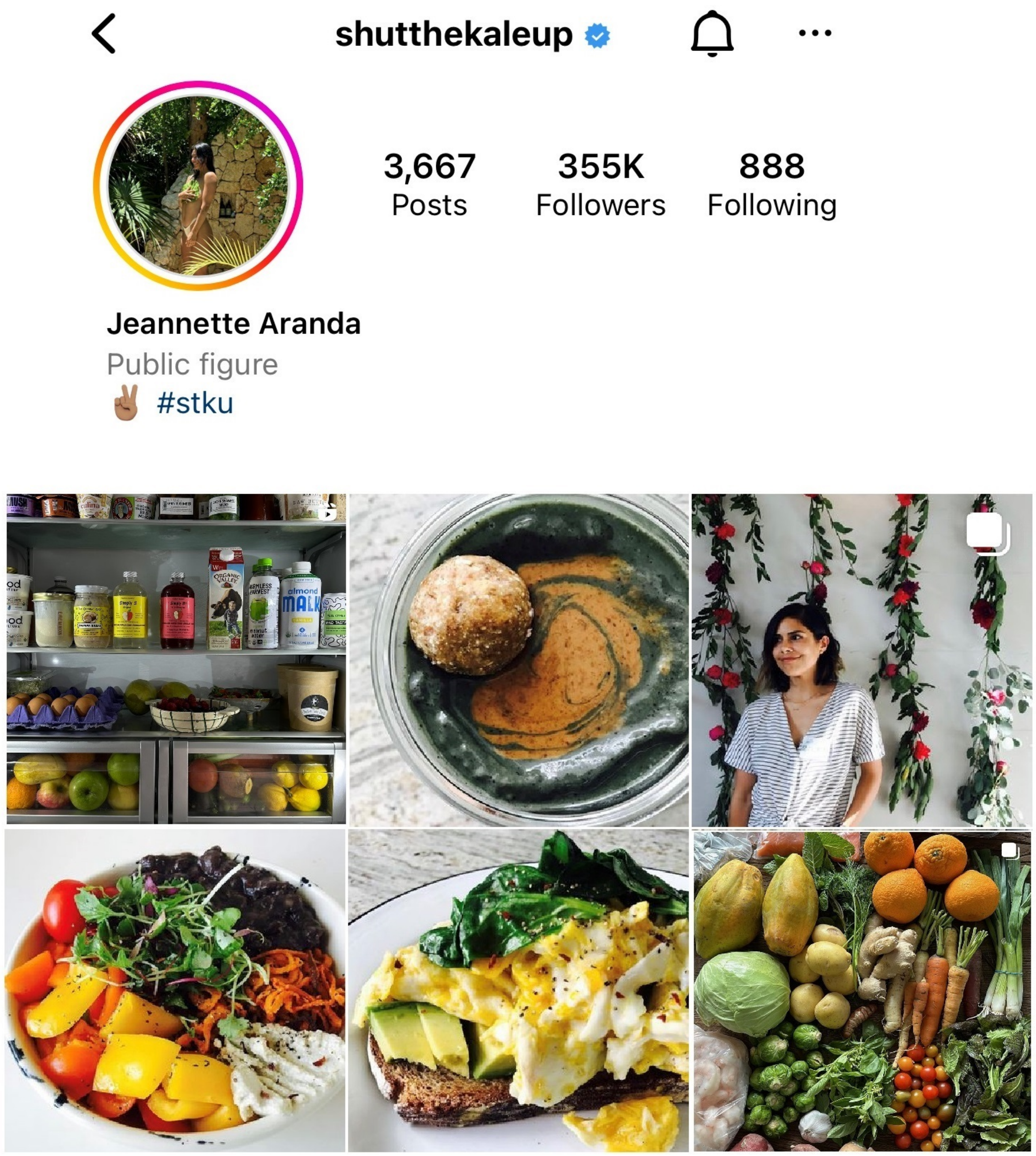
**Abstract**

Appendix A contains examples of sponsored Instagram posts. Appendix B presents formal proofs of all results.

# Appendix A: Influencer Examples

**Shutthekaleup Instagram Content (Non-sponsored Posts)**

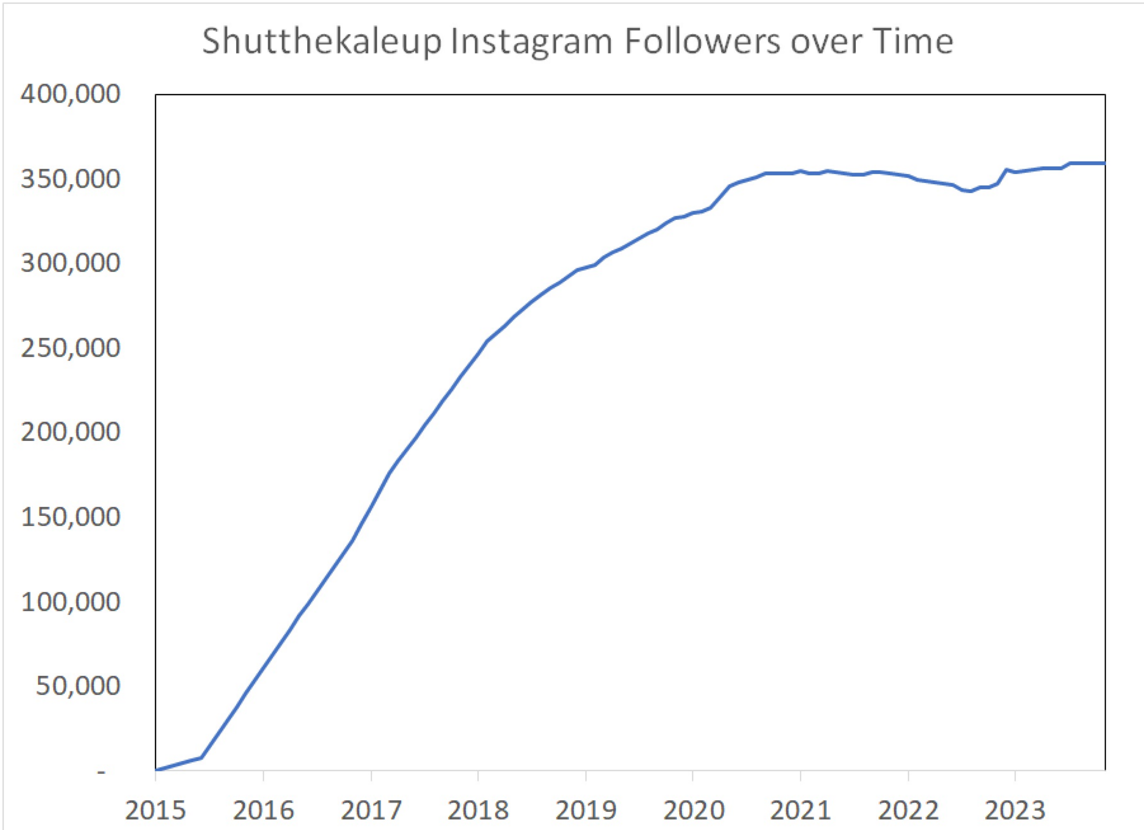


## Shutthekaleup Food Endorsements (posted in 2017)


**Shutthekaleup Non-food Endorsements (posted in 2018 and 2023)**

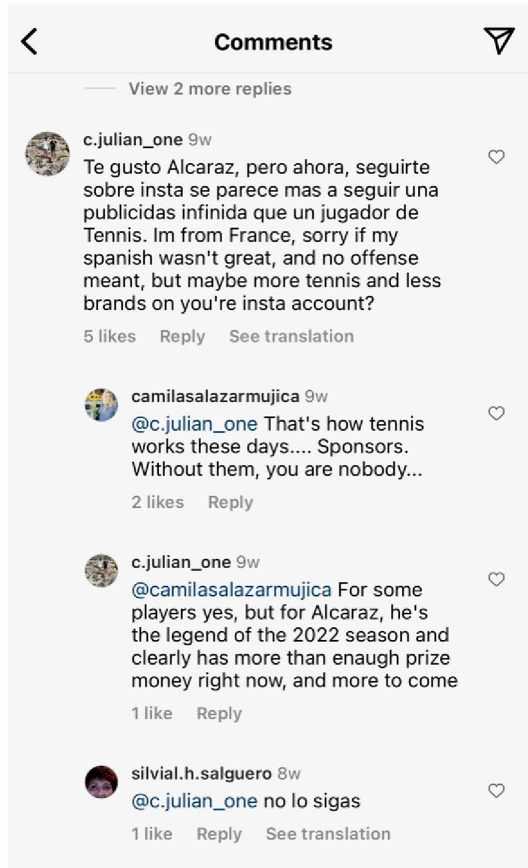


Retrieved May 14, 2023, https://www.instagram.com/shutthekaleup/




Data Sources: Shutthekaleup's first Instagram post (1/4/2015), Wayback Machine (5/31/2015 and 4/30/2016), Marina Elaine Gunn blog post (11/15/2016), Natasha Cipriani blog post (3/5/2017), *Forbes* article (2/9/2018), NotJustAnalytics.com (monthly data from 6/1/2018 to 11/1/2023)

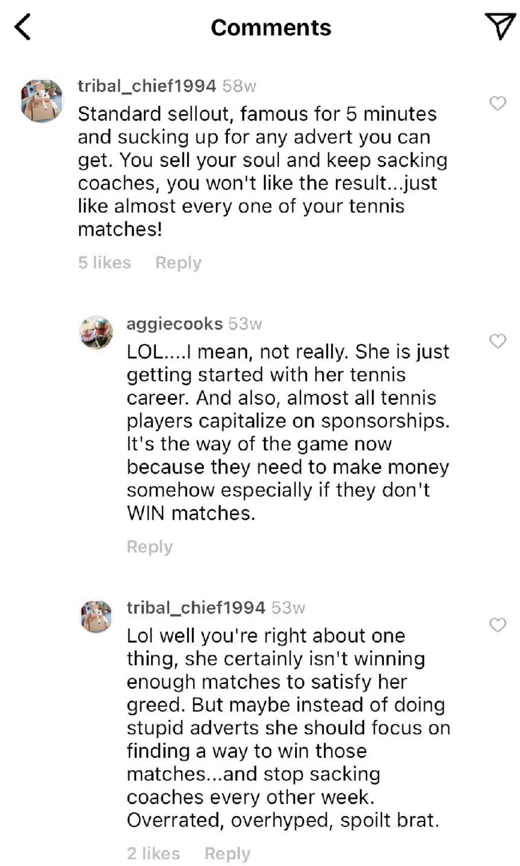
## Carlos Alcaraz BMW Endorsement (posted in 2023)


(The last comment in Spanish: "Do not follow him.")

Retrieved May 19, 2023, https://www.instagram.com/carlitosalcarazz/

## Emma Raducanu Porsche Endorsement (posted in 2022)


Retrieved June 12, 2023, https://www.instagram.com/emmaraducanu/

# Appendix B: Proofs

### Proof of Lemma 1

We will compute an upper bound on the difference between $V(A_t + \epsilon)$ and $V(A_t)$ when $A_t > \frac{1}{2}$. For $u > t$, define $\widehat{A}_u$ and $\overline{A}_u$ as the awareness at time $u$ if the influencer starts at time $t$ with awareness $A_t + \epsilon$ and $A_t$, respectively, and follows the policy that would be optimal starting with awareness $A_t + \epsilon$. For any given policy, awareness is strictly increasing over time, which ensures $\widehat{A}_u > \overline{A}_u$ for all $u > t$. Furthermore, the growth rate is decreasing in awareness for $A_t > \frac{1}{2}$, which implies $\widehat{A}_u - \overline{A}_u$ decreases over time. Because the initial difference in awareness is $\epsilon$, $\widehat{A}_u - \overline{A}_u < \epsilon$ for all $u > t$. Because $\pi_u$ equals either $A_u$ or $\gamma A_u$, the value increase from permanently increasing awareness by $\epsilon$ would be less than or equal to $\frac{\epsilon}{r}$, so the actual value increase from increasing awareness by $\epsilon$ is less than this amount. QED

### Proof of Lemma 2

Being inauthentic at time $t$ increases instantaneous profits by $(\gamma\phi - 1)A_t$. Being authentic at time $t$ increases the growth rate of awareness by $(1 - \gamma)\beta A_t(1 - A_t)$. Lemma 1 guarantees the first derivative of the value function with respect to awareness is less than $\frac{1}{r}$ for $A_t > \frac{1}{2}$. Therefore, the influencer prefers to be inauthentic if $A_t > \frac{1}{2}$ and $(\gamma\phi - 1) > (1 - \gamma)\beta(1 - A_t)\frac{1}{r}$. QED

### Proof of Lemma 3

If the influencer is always inauthentic starting at time $t$, then for time $u \geq t$, awareness grows according to $\frac{dA_u}{du} = \gamma\beta A_u(1 - A_u)$. This differential equation has the following solution:

$$A_u = \frac{1}{1 + \left(\frac{1-A_t}{A_t}\right) e^{-\gamma\beta(u-t)}} \tag{14}$$

$$\frac{dA_u}{du} = \frac{\gamma\beta\left(\frac{1-A_t}{A_t}\right) e^{-\gamma\beta(u-t)}}{\left[1 + \left(\frac{1-A_t}{A_t}\right) e^{-\gamma\beta(u-t)}\right]^2} \tag{15}$$

To verify this solution, one can differentiate (14) and check that (15) is the derivative, and note these equations also satisfy $\frac{dA_u}{du} = \gamma\beta A_u(1 - A_u)$. Finally, if we set $u = t$, we can confirm that (14) equals $A_t$.

If the influencer is inauthentic, then $\pi_u = \gamma\phi A_u$. Therefore, the value function is:

$$\underline{V}(A_t) = \int_{u=t}^{\infty} \frac{e^{-r(u-t)}\gamma\phi}{\left[1 + \left(\frac{1-A_t}{A_t}\right) e^{-\gamma\beta(u-t)}\right]} du \tag{16}$$

QED

**Proof of Proposition 1**

If $\gamma\phi < 1$, being authentic leads to higher current profits and faster growth, so the influencer is always authentic.

If $\gamma\phi > 1$, Lemma 2 guarantees the influencer eventually becomes inauthentic. However, the derivations in the body of the paper show that, if $A_0$ is sufficiently small and Condition 1 holds, the influencer cannot always be inauthentic starting at awareness $A_0$ because she would prefer to deviate from this policy and to be authentic for a period starting at time zero.

The only remaining step is to show the influencer may start by being authentic to grow quickly and then switch to being inauthentic to generate more profits, but can never change policies in the other direction and go from being inauthentic to authentic.

Define $A^*$ as the awareness level that solves $(1-\gamma)\beta(1-A_t)\frac{d\underline{V}(A_t)}{dA_t} = (\gamma\phi - 1)$ for $A_t = A^*$. When this equation holds, the additional value of faster growth from being authentic exactly equals the value of greater immediate profits from being inauthentic. The derivative of the value function if the influencer is always inauthentic, denoted by $\frac{d\underline{V}(A_t)}{dA_t}$ in equation (3), is decreasing in $A_t$. Therefore, for all $A_t > A^*$, we have $(1-\gamma)\beta(1-A_t)\frac{d\underline{V}(A_t)}{dA_t} < (\gamma\phi-1)$, which implies the influencer always stays inauthentic for awareness levels higher than $A^*$.

We now show the influencer is authentic for all awareness levels below $A^*$. Suppose the influencer is authentic starting at time $t$ and then switches to being inauthentic at time $u^*$, which is chosen as the time at which $A_t = A^*$. Similar derivations to those in the proof of Lemma 3 show that, on the time interval $u \in [t, u^*]$, awareness is given by $A_u = \frac{1}{1+\left(\frac{1-A_t}{A_t}\right)e^{-\beta(u-t)}}$. The value function is then:

$$V(A_t) = \int_{u=t}^{u^*} \frac{e^{-r(u-t)}}{\left[1+\left(\frac{1-A_t}{A_t}\right)e^{-\beta(u-t)}\right]}\, du + e^{-r(u^*-t)}\underline{V}(A_{u^*}) \tag{17}$$

When differentiating this value function, the envelope theorem implies that the change in the optimal $u^*$ has only a second-order effect, so we can simply differentiate each component. Taking second derivatives shows that the profits on the interval $[t, u^*]$ are concave in $A_t$, and $A_{u^*}$ is also concave in $A_t$. We have already shown that the function $\underline{V}$ is concave. Therefore, the second derivative of $V(A_t)$ is negative.

Finally, if it is optimal to be authentic for awareness level $A_t$, that implies $(1-\gamma)\beta(1-A_t)V'(A_t) > (\gamma\phi - 1)$. Given that $V'(A_t)$ is decreasing in $A_t$, this inequality must also hold for all awareness levels less than $A_t$, so it must also be optimal to be authentic at lower levels of awareness. Thus, if it is ever optimal to be authentic, then the optimal policy is to be authentic starting at awareness level $A_0$ and then switch to being inauthentic at the awareness level stated in the proposition. QED

**Proof of Lemma 4**

We will compare $V(A_t + \epsilon)$ with $V(A_t)$ for small $\epsilon$ and show that the difference $V(A_t + \epsilon) - V(A_t)$ is decreasing in $\alpha$.

For $u > t$, let $\widehat{A}_u$ and $\overline{A}_u$ denote awareness at time $u$ if the influencer starts with awareness $A_t + \epsilon$ and $A_t$, respectively, and follows the policy that is optimal starting with awareness $A_t$. For sufficiently small $\epsilon$, the difference in the optimal policy starting at $A_t + \epsilon$ versus $A_t$ has only a second order effect, and the envelope theorem implies we can perform comparative statics on the difference in profits from these starting points using the same policy.

If the influencer is authentic at time $u$, the equation of motion is $\frac{dA_u}{du} = (\alpha + \beta A_u)(1 - A_u)$, so the rate of change in the difference between $\widehat{A}_u$ and $\overline{A}_u$ is given by:

$$\frac{d\widehat{A}_u}{du} - \frac{d\overline{A}_u}{du} = (\beta - \alpha)(\widehat{A}_u - \overline{A}_u) - \beta(\widehat{A}_u^2 - \overline{A}_u^2) \tag{18}$$

$$= (\beta - \alpha)(\widehat{A}_u - \overline{A}_u) - \beta((\overline{A}_u + (\widehat{A}_u - \overline{A}_u))^2 - \overline{A}_u^2) \tag{19}$$

$$= (\beta - \alpha)(\widehat{A}_u - \overline{A}_u) - \beta(2\overline{A}_u(\widehat{A}_u - \overline{A}_u) + (\widehat{A}_u - \overline{A}_u)^2) \tag{20}$$

If the influencer is inauthentic, each term $\beta$ is replaced by $\gamma\beta$. For a given difference $\widehat{A}_u - \overline{A}_u$, the first term in this equation is decreasing in $\alpha$. Furthermore, for a given policy, the value of $\overline{A}_u$ increases with $\alpha$ due to faster growth, so the second term in the equation also is decreasing in $\alpha$ for a given value of $\widehat{A}_u - \overline{A}_u$. Thus, for all $u > t$ the resulting gap in future awareness and future profits based on an $\epsilon$ increase in awareness at time $t$ is decreasing in $\alpha$. Furthermore, the envelope theorem implies the effect of an increase in $\alpha$ on the optimal policy has only a second order effect on the value function, so the derivative of the value function with respect to awareness is also decreasing in $\alpha$. QED

**Proof of Proposition 2**

As in the main version of the model, it is optimal for the influencer to be inauthentic if $(\gamma\phi - 1) > (1-\gamma)\beta(1-A_t)V'(A_t)$. Lemma 4 shows that $V'(A_t)$ is decreasing in $\alpha$. Therefore, when $\alpha$ increases, the right side of this inequality decreases, which causes the influencer to become inauthentic at a lower level of awareness. QED

**Proof of Lemma 5**

If the influencer is always authentic starting at time $t$, then for time $u \geq t$, awareness grows according to $\frac{dA_u}{du} = \beta A_u(1-A_u) - xA_u$. This differential equation has the following solution:

$$A_u = \frac{1}{\left[\frac{\beta}{\beta-x} + \left(\frac{1-A_t\frac{\beta}{\beta-x}}{A_t}\right)e^{-(\beta-x)(u-t)}\right]} \tag{21}$$

$$\frac{dA_u}{du} = \frac{(\beta-x)\left(\frac{1-A_t\frac{\beta}{\beta-x}}{A_t}\right)e^{-(\beta-x)(u-t)}}{\left[\frac{\beta}{\beta-x} + \left(\frac{1-A_t\frac{\beta}{\beta-x}}{A_t}\right)e^{-(\beta-x)(u-t)}\right]^2} \tag{22}$$

To verify this solution, one can differentiate (21) and check that (22) is the derivative, and note these equations also satisfy $\frac{dA_u}{du} = \beta A_u(1-A_u) - xA_u$. Finally, if we set $u = t$, we can confirm that (21) equals $A_t$.

If the influencer is authentic, then $\pi_u = A_u$. Therefore, the value function is:

$$\overline{V}(A_t) = \int_{u=t}^{\infty} \frac{e^{-r(u-t)}}{\left[\frac{\beta}{\beta-x} + \left(\frac{1-A_t\frac{\beta}{\beta-x}}{A_t}\right)e^{-(\beta-x)(u-t)}\right]} \, du \tag{23}$$

QED

## Proof of Proposition 3

We will show Condition 2 ensures the influencer never has an incentive to deviate from the policy of being authentic.

To compute the marginal value of awareness given a policy of authenticity, we differentiate the value function (23) with respect to awareness:

$$\frac{d\overline{V}(A_t)}{dA_t} = \int_{u=t}^{\infty} \frac{e^{-(r+\beta-x)(u-t)}}{\left[A_t\frac{\beta}{\beta-x} + \left(1 - A_t\frac{\beta}{\beta-x}\right)e^{-(\beta-x)(u-t)}\right]^2}\,du \tag{24}$$

The influencer has an incentive to maintain her policy of authenticity at awareness level $A_t$ if:

$$(1-\gamma)\beta(A_t)(1-A_t)\frac{d\overline{V}(A_t)}{dA_t} > (\gamma\phi - 1)A_t \tag{25}$$

We first show the influencer stays authentic at the maximum feasible awareness level $A_t = \frac{\beta-x}{\beta}$. Evaluating (24) at this level of awareness, we have $\frac{d\overline{V}(A_t)}{dA_t} = \frac{1}{r+\beta-x}$. Furthermore, $1 - A_t = \frac{x}{\beta}$. Inserting these values into (25), we find this inequality is equivalent to Condition 2, so the influencer stays authentic at the maximum feasible awareness if this condition holds.

Because $(1 - A_t)$ and $\frac{d\overline{V}(A_t)}{dA_t}$ are both decreasing in $A_t$, Condition 2 ensures (25) holds and the influencer also stays authentic for all lower levels of awareness. QED

## Proof of Proposition 4

As shown in the body of the paper, as $A_t \to 0$, the increase in profits from being inauthentic converges to $vz(\phi - 1)$, and the increase in rate of viral posts from being inauthentic converges to $\mu(1-\theta)v$. We now need to show that the value of the increased growth rate from being authentic, given by $(1-\gamma)\beta A_t(1-A_t)\frac{dV(A_t)}{dA_t}$, approaches zero as $A_t \to 0$. To see why this is true, note that the growth rate of awareness based on current followers is bounded below $\beta A_t$ for all $A_t$. Therefore, for any positive value $\Delta$, the increase in instantaneous profits based on an increase in awareness of size $\epsilon$ remains less than $\Delta$ for a length of time that diverges to infinity as $\epsilon \to 0$, which implies that the value of this awareness increase approaches zero as $\epsilon \to 0$. Thus, as $A_t \to 0$, we have $A_t \frac{dV(A_t)}{dA_t} \to 0$. We have shown that the effect on profits from being inauthentic converges to $vz(\phi - 1)$ whereas the value of faster growth from being authentic converges to zero as awareness approaches zero, which implies the influencer is inauthentic for sufficiently small awareness.

As $A_t \to 1$, the effect of authenticity on awareness growth approaches zero, the effect of a viral post on awareness also approaches zero, and the effect of being inauthentic on current profits converges to $\gamma\phi + vz\phi(1-\gamma) - 1$, so the influencer is inauthentic if this effect on profits is positive an authentic otherwise. QED

**Proof of Proposition 5**

As $A_{c,t} \to 1$, the effect of being inauthentic on current profits from the core segment approaches $\gamma\phi_c - 1$. As $A_{m,t} \to 1$, the effect of being inauthentic on current profits from the mainstream segment approaches $\gamma\phi_m - 1$. For both segments, the effect of authenticity on awareness growth eventually approaches zero. Therefore, for sufficiently high awareness with both segments, the influencer is inauthentic if $\gamma\phi_c + \gamma\phi_m > 2$ and authentic otherwise. QED

**Proof of Proposition 6**

The effect on profits from endorsing a product with bad fit is $\omega\widehat{U}F_t$. For the influencer to commit to reject such an offer, the reduction in value from moving to the bad equilibrium must exceed these profits. The value of staying with the optimal policy is greater than or equal to the value of staying with a policy of always being authentic, which we denote by $\overline{V}(A_t)$. Denote the value of the bad equilibrium by $\widehat{V}(A_t)$. For

$u > t$, let $\overline{A}_u$ denote awareness at time $u$ given the policy of always being authentic starting at time $t$, and let $\widehat{A}_u$ denote awareness at time $u$ if the influencer moves to the bad equilibrium and begins endorsing all products at time $t$. The profit difference between these policies at time $u$ is $\overline{A}_u - \widehat{\gamma}\widehat{\phi}\widehat{A}_u$. Awareness grows over time under both policies and is always greater under the policy of being authentic, which implies $\overline{A}_u - \widehat{\gamma}\widehat{\phi}\widehat{A}_u > (1 - \widehat{\gamma}\widehat{\phi})A_t$. The value of permanently increasing profits by $(1 - \widehat{\gamma}\widehat{\phi})A_t$ is $\frac{1}{r}$ times this profit difference. Therefore, given the condition of the proposition, $\frac{1-\widehat{\gamma}\widehat{\phi}}{r} > \omega\widehat{U}$, the value increase from staying in the good equilibrium exceeds the profits from endorsing a product with bad fit, and the influencer can commit to the optimal policy in equilibrium. QED